About amenability of subgroups of the group of diffeomorphisms of the interval.

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Averaging linear functional on the space continuous functions of the group of diffeomorphisms of interval is found. Amenability of several discrete subgroups of the group of diffeomorphisms $\mathrm{Diff}^3([0,1])$ of interval is prove. In particular, a solution of the problem of amenability of the Thompson's group F is given.

1. The main result.

Let $\mathrm{Diff}^1_+([0,1])$ be the group of all diffeomorphisms of class C^1 of interval $[0,1]$ that preserve the endpoints of interval, and let $\mathrm{Diff}^3_+([0,1])$ be the subgroup of $\mathrm{Diff}^1_+([0,1])$ consisting of all diffeomorphisms of class $C^3([0,1])$, and

$$\mathrm{Diff}^3_0([0,1]) = \{f \in \mathrm{Diff}^3_+([0,1]) : f'(0) = f'(1) = 1\}.$$

The group $\mathrm{Diff}^1_+([0,1])$ is equipped with the topology inherited from the space $C^1([0,1])$.

Let us denote $D_n = \{(x_1, ..., x_{n-1}) : 0 < x_1 < ... < x_{n-1} < 1\} \subset \mathbf{R}^{n-1}$ and $x_0 = 0,\ x_n = 1$.

We say that a subgroup G of $\mathrm{Diff}^3_0([0,1])$ satisfies condition (a), if

(i) there are a integer $l_n > l_{n-1}\ (l_0 = 1)$ for any natural n and a countably additive Borel measure η_n on D_{l_n} such that $\eta_n(D_{l_n}) = 1$,

(ii) for any positive ε and any $g \in G$, we can find natural $N(\varepsilon, g)$ such that, for any $n > N(\varepsilon, g)$, it exists a Borel subset $Z_{n,\varepsilon,g} \subset D_{l_n}$ that $\eta_n(Z_{n,\varepsilon,g}) > 1 - \varepsilon$ and $\max\limits_{1\le k\le l_n} (x_k - x_{k-1}) < \varepsilon$ for any $(x_1, x_2, ..., x_{l_n-1}) \in Z_{n,\varepsilon,g}$ where $x_0 = 0,\ x_{l_n} = 1$,

(iii) $(1-\varepsilon)\eta_n(Y) < \eta_n(gY) < (1+\varepsilon)\eta_n(Y)$ for any Borel subset $Y \subset Z_{n,\varepsilon,g}$ where $gY = \{(g(x_1), g(x_2), ..., g(x_{l_n-1})) : (x_1, x_2, ..., x_{l_n-1}) \in Y\}$.

For any positive $\delta < 1$, denote by $C^{1,\delta}_0([0,1])$ the set of all functions $f \in C^1([0,1])$ such that $f(0) = 0$ and $\exists C > 0\ \ \forall t_1, t_2 \in [0,1]\ \ |f'(t_2) - f'(t_1)| < C|t_2 - t_1|^\delta$. Define a Banach structure on the linear space $C^{1,\delta}_0([0,1])$ by a norm

$$\|f\|_{1,\delta} = |f'(0)| + \sup_{t_1,t_2\in[0,1]} \frac{|f'(t_2) - f'(t_1)|}{|t_2 - t_1|^\delta}$$

for any function $f \in C^{1,\delta}_0([0,1])$.

Let $\mathrm{Diff}^{1,\delta}_+([0,1]) = \mathrm{Diff}^1_+([0,1]) \bigcap C^{1,\delta}_0([0,1])$. It is easy to see that $\mathrm{Diff}^{1,\delta}_+([0,1])$ is a subgroup of the group $\mathrm{Diff}^1_+([0,1])$. The subgroup $\mathrm{Diff}^{1,\delta}_+([0,1])$ is equipped with the topology inherited from the space $C^{1,\delta}_0([0,1])$.

Let $C_b(\mathrm{Diff}^{1,\delta}_+([0,1]))$ be the linear space of all bounded continuous functions on the space $\mathrm{Diff}^{1,\delta}_+([0,1])$, and let $C_b(\mathrm{Diff}^1_+([0,1]))$ be the linear space of all bounded continuous functions on the space $\mathrm{Diff}^1_+([0,1])$.

Introduce the functions $e_{1,\delta} : \mathrm{Diff}^{1,\delta}_+([0,1]) \to \mathbf{R},\quad e_{1,0} : \mathrm{Diff}^1_+([0,1]) \to \mathbf{R}$ by setting $e_{1,\delta}(g) = 1$ for any $g \in \mathrm{Diff}^{1,\delta}_+([0,1])$ and $e_{1,0}(f) = 1$ for any $f \in \mathrm{Diff}^1_+([0,1])$. Let $F_g(f) = F(g^{-1} \circ f)$ for any $g \in \mathrm{Diff}^3_0([0,1])$, $f \in \mathrm{Diff}^{1,\delta}_+([0,1])$ and $F \in C_b(\mathrm{Diff}^{1,\delta}_+([0,1]))$.

Theorem 1. *If a subgroup G of* $\mathrm{Diff}^3_0([0,1])$ *satisfies condition* (a) *and a positive* $\delta < \frac{1}{2}$ *then there exists a linear functional* $L_\delta : C_b(\mathrm{Diff}^{1,\delta}_+([0,1])) \to \mathbf{R}$ *such that* $L_\delta(e_{1,\delta}) = 1$, $|L_\delta(F)| \leq \sup_{f \in \mathrm{Diff}^{1,\delta}_+([0,1])} |F(f)|$, $L_\delta(F) \geq 0$ *for any nonnegative function* $F \in C_b(\mathrm{Diff}^{1,\delta}_+([0,1]))$, *and* $L_\delta(F_g) = L_\delta(F)$ *for any* $g \in G$ *and* $F \in C_b(\mathrm{Diff}^{1,\delta}_+([0,1]))$.

The restriction of any function of the space $C_b(\mathrm{Diff}^1_+([0,1]))$ on $\mathrm{Diff}^{1,\delta}_+([0,1])$ belongs to the space $C_b(\mathrm{Diff}^{1,\delta}_+([0,1]))$. Hence we obtain the following assertion.

Corollary 1.1. *If a subgroup G of* $\mathrm{Diff}^3_0([0,1])$ *satisfies condition* (a) *then there exists a linear functional* $L_0 : C_b(\mathrm{Diff}^1_+([0,1])) \to \mathbf{R}$ *such that* $L_0(e_{1,0}) = 1$, $|L_0(F)| \leq \sup_{f \in \mathrm{Diff}^1_+([0,1])} |F(f)|$, $L_0(F) \geq 0$ *for any nonnegative function* $F \in C_b(\mathrm{Diff}^1_+([0,1]))$, *and* $L_0(F_g) = L_0(F)$ *for any* $g \in G$ *and* $F \in C_b(\mathrm{Diff}^1_+([0,1]))$.

We say that a discrete subgroup G of $\mathrm{Diff}^3_0([0,1])$ satisfies condition (b), if there is a such $C > 0$ that $\sup_{t \in [0,1]} |\ln(g_1'(t)) - \ln(g_2'(t))| \geq C$ for any $g_1, g_2 \in G, g_1 \neq g_2$.

Theorem 2. *If a discrete subgroup G of* $\mathrm{Diff}^3_0([0,1])$ *satisfies conditions* $(a), (b)$, *then the subgroup G is amenable.*

In [2] È.Ghys and V.Sergiescu proved that the Thompson's group F is isomorphic to a discrete subgroup G of $\mathrm{Diff}^3_0([0,1])$ which satisfies condition (b).

Corollary 2.1. *The Thompson's group F is amenable.* .

2. Proof of Theorem 1.

Define the mapping $A : \mathrm{Diff}^1_+([0,1]) \to C_0([0,1])$ by setting

$$A(q)(t) = \ln(q'(t)) - \ln(q'(0)) \qquad \forall t \in [0,1].$$

The mapping A is a topological isomorphism between the space $\mathrm{Diff}^1_+([0,1])$, $C_0([0,1])$ moreover

$$A^{-1}(\xi)(t) = \frac{\int_0^t e^{\xi(\tau)} d\tau}{\int_0^1 e^{\xi(\tau)} d\tau}.$$

Introduce the Wiener measure w on the space $C_0([0,1])$. Define a Borel measure ν on $\mathrm{Diff}^1_+([0,1])$ by setting $\nu(X) = w(A(X))$ for any Borel subset X of topological space $\mathrm{Diff}^1_+([0,1])$.

Let $\delta \in (0, \frac{1}{2})$. It follows from the properties of Wiener measure w (see [4]) that measure ν is concentrated on the set $E_\delta = \mathrm{Diff}^{1,\delta}_+([0,1])$, i.d. $\nu(E_\delta) = 1$, moreover the Borel subsets of metric space E_δ is measurable with respect to the measure ν.

As it was proved in [3], the measure ν is quasi-invariant with respect to the left action of subgroup $\mathrm{Diff}^3_+([0,1])$ on the group $\mathrm{Diff}^1_+([0,1])$, moreover

$$\nu(gX) = \frac{1}{\sqrt{g'(0)g'(1)}} \int_X e^{\frac{g''(0)}{g'(0)} q'(0) - \frac{g''(1)}{g'(1)} q'(1) + \int_0^1 S_g(q(t))(q'(t))^2 dt} \nu(dq),$$

for any Borel subset X of topological space $\mathrm{Diff}^1_+([0,1])$,and any $g \in \mathrm{Diff}^3_+([0,1])$, where $gX = \{g \circ q : q \in X\}$ and $S_g(\tau) = \frac{g'''(\tau)}{g'(\tau)} - \frac{3}{2}(\frac{g''(\tau)}{g'(\tau)})^2$ (the Schwartz derivative of function g).

For the proof of Theorem 1 we need the following auxiliary assertions

Lemma 1. *The following equality is valid*

$$\int\limits_{E_\delta} (q'(0))^l \, \nu(dq) = \int\limits_{E_\delta} (q'(1))^l \, \nu(dq)$$

for any natural l.

Proof. Let $\xi = A(q)$, i.d. $\xi(t) = \ln(q'(t)) - \ln(q'(0))$. Then

$$q'(0) = \frac{1}{\int_0^1 e^{\xi(\tau)} d\tau}, \; q'(1) = \frac{e^{\xi(1)}}{\int_0^1 e^{\xi(\tau)} d\tau}.$$

Let us take

$$M_l = \int\limits_{E_\delta} (q'(1))^l \, \nu(dq) = \int\limits_{\mathrm{Diff}^1_+([0,1])} (q'(1))^l \, \nu(dq) =$$

$$= \int\limits_{C_0([0,1])} \left(\frac{e^{\xi(1)}}{\int_0^1 e^{\xi(\tau)} d\tau}\right)^l w(d\xi) = \int\limits_{C_0([0,1])} \left(\frac{1}{\int_0^1 e^{\xi(1-\tau)-\xi(1)} d\tau}\right)^l w(d\xi)$$

Let $\zeta(t) = \xi(1-t) - \xi(1)$. The Wiener measure w is invariant with respect to the action $\zeta \longmapsto \xi$, thus,

$$M_l = \int\limits_{C_0([0,1])} \left(\frac{1}{\int_0^1 e^{\zeta(\tau)} d\tau}\right)^l w(d\zeta) =$$

$$= \int\limits_{\mathrm{Diff}^1_+([0,1])} (q'(0))^l \, \nu(dq) = \int\limits_{E_\delta} (q'(0))^l \, \nu(dq),$$

which implies the assertion of Lemma 1.

Introduce the measure $\nu_n = \nu \otimes ... \otimes \nu$ on the space $E^n_\delta = E_\delta \times ... \times E_\delta$.

Let $c_1 = 1 + M_1 + M_2 + \int\limits_{E_\delta} (\int_0^1 (q'(t))^2 dt) \, \nu(dq)$.

For any $r > 0$, $g \in \mathrm{Diff}^3_+([0,1])$, $\overline{x} = (x_1, ..., x_{n-1}) \in D_n$, we write $C_g = 1 + \max\limits_{0 \le t \le 1} (|\frac{g''(t)}{g'(t)}| + (\frac{g''(t)}{g'(t)})^2 + |\frac{g'''(t)}{g'(t)}|)$ and

$$X_{r,g,\overline{x}} = \{(q_1, ..., q_n) : \; q_1, ..., q_n \in E_\delta$$

$$|\sum_{k=1}^n [(x_k - x_{k-1})(\frac{g''(x_{k-1})}{g'(x_{k-1})} q'_k(0) - \frac{g''(x_k)}{g'(x_k)} q'_k(1)) +$$

$$+ (x_k - x_{k-1})^2 \int_0^1 S_g(x_{k-1} + (x_k - x_{k-1}) q_k(t))(q'_k(t))^2 dt]| \le 4 c_1 C_g r\}.$$

Lemma 2. *If $\epsilon \in (0,1)$, then the following inequality is fulfilled $\nu_n(E^n_\delta \setminus X_{\sqrt[3]{\epsilon},g,\overline{x}}) \le 2\sqrt[3]{\epsilon}$ for any $g \in \mathrm{Diff}^3_+([0,1])$, for any positive integer n and $\overline{x} = (x_1, ..., x_{n-1}) \in D_n$, satisfying the inequality $\max\limits_{1 \le k \le n} (x_k - x_{k-1}) < \epsilon$.*

Proof. Let

$$f_1(q_1, ..., q_n) = \sum_{k=1}^n (x_k - x_{k-1})(\frac{g''(x_{k-1})}{g'(x_{k-1})} q'_k(0) - \frac{g''(x_k)}{g'(x_k)} q'_k(1)).$$

Then

$$I_1 = \int\limits_{E_\delta} ... \int\limits_{E_\delta} f_1(q_1, ..., q_n)\, \nu(dq_1)...\nu(dq_n) = M_1 \sum_{k=1}^{n} (x_k - x_{k-1})(\frac{g''(x_{k-1})}{g'(x_{k-1})} - \frac{g''(x_k)}{g'(x_k)}).$$

As $|\frac{g''(x_{k-1})}{g'(x_{k-1})} - \frac{g''(x_k)}{g'(x_k)}| \leq C_g(x_k - x_{k-1})$, we have

$$|I_1| \leq M_1 C_g \sum_{k=1}^{n} (x_k - x_{k-1})^2 \leq M_1 C_g \epsilon \sum_{k=1}^{n} (x_k - x_{k-1}) = M_1 C_g \epsilon.$$

If $k \neq l$ then

$$\int\limits_{E_\delta} \int\limits_{E_\delta} (\frac{g''(x_{k-1})}{g'(x_{k-1})}(q'_k(0) - M_1) - \frac{g''(x_k)}{g'(x_k)}(q'_k(1) - M_1))$$

$$(\frac{g''(x_{l-1})}{g'(x_{l-1})}(q'_l(0) - M_1) - \frac{g''(x_l)}{g'(x_l)}(q'_l(1) - M_1))\nu(dq_k)\nu(dq_l) = 0,$$

therefore

$$I_2 = \int\limits_{E_\delta} ... \int\limits_{E_\delta} (f_1(q_1, ..., q_n) - I_1)^2\, \nu(dq_1)...\nu(dq_n) =$$

$$\sum_{k=1}^{n} (x_k - x_{k-1})^2 \int\limits_{E_\delta} [\frac{g''(x_{k-1})}{g'(x_{k-1})}(q'_k(0) - M_1) - \frac{g''(x_k)}{g'(x_k)}(q'_k(1) - M_1)]^2 \nu(dq_k) \leq$$

$$\leq 2 \sum_{k=1}^{n} (x_k - x_{k-1})^2 [(\frac{g''(x_{k-1})}{g'(x_{k-1})})^2 \int\limits_{E_\delta} (q'_k(0) - M_1)^2 \nu(dq_k) +$$

$$+ (\frac{g''(x_k)}{g'(x_k)})^2 \int\limits_{E_\delta} (q'_k(1) - M_1)^2 \nu(dq_k)] =$$

$$= 2 \sum_{k=1}^{n} (x_k - x_{k-1})^2 [(\frac{g''(x_{k-1})}{g'(x_{k-1})})^2 + (\frac{g''(x_k)}{g'(x_k)})^2][\int\limits_{E_\delta} (q'_k(0))^2 \nu(dq_k) - (M_1)^2] \leq$$

$$\leq 4 M_2 C_g \sum_{k=1}^{n} (x_k - x_{k-1})^2 \leq 4 M_2 C_g \epsilon \sum_{k=1}^{n} (x_k - x_{k-1}) = 4 M_2 C_g \epsilon.$$

Hence,

$$\nu_n(\{(q_1, ..., q_n) : |f_1(q_1, ..., q_n) - I_1| \geq 2c_4 C_g \sqrt[3]{\epsilon}\}) \leq \frac{I_2}{(2c_4 C_g \sqrt[3]{\epsilon})^2} \leq \frac{4 M_2 C_g \epsilon}{(2c_4 C_g \sqrt[3]{\epsilon})^2} \leq \sqrt[3]{\epsilon}.$$

Thus

$$\nu_n(\{(q_1, ..., q_n) : |f_1(q_1, ..., q_n)| \geq 3c_1 C_g \sqrt[3]{\epsilon}\}) \leq \sqrt[3]{\epsilon}.$$

Let

$$f_2(q_1, ..., q_n) = \sum_{k=1}^{n} ((x_k - x_{k-1})^2 \int_0^1 S_g(x_{k-1} + (x_k - x_{k-1})q_k(t))(q'_k(t))^2 dt.$$

Then

$$I_3 = \int\limits_{E_\delta} ... \int\limits_{E_\delta} |f_2(q_1, ..., q_n)|\, \nu(dq_1)...\nu(dq_n) \leq$$

$$\leq 2C_g \sum_{k=1}^n (x_k - x_{k-1})^2 \int_{E_\delta} (\int_0^1 (q_k'(t))^2 dt) \nu(dq_k) \leq$$

$$\leq 2c_1 C_g \epsilon \sum_{k=1}^n (x_k - x_{k-1}) = 2c_1 C_g \epsilon.$$

Thus

$$\nu_n(\{(q_1,...,q_n) : |f_2(q_1,...,q_n)| \geq 2c_1 C_g \sqrt[3]{\epsilon}\}) \leq \frac{I_3}{2c_1 C_g \sqrt[3]{\epsilon}} \leq \frac{2c_1 C_g \epsilon}{2c_1 C_g \sqrt[3]{\epsilon}} = (\sqrt[3]{\epsilon})^2 \leq \sqrt[3]{\epsilon}.$$

Hence,

$$\nu_n(E_\delta^n \setminus X_{\sqrt[3]{\epsilon},g,\overline{x}}) =$$
$$= \nu_n(\{(q_1,...,q_n) : |f_1(q_1,...,q_n) + f_2(q_1,...,q_n)| \geq 4c_1 C_g \sqrt[3]{\epsilon}\}) \leq$$
$$\leq \nu_n(\{(q_1,...,q_n) : |f_1(q_1,...,q_n)| \geq 2c_1 C_g \sqrt[3]{\epsilon}\}) +$$
$$+ \nu_n(\{(q_1,...,q_n) : |f_2(q_1,...,q_n)| \geq 2c_1 C_g \sqrt[3]{\epsilon}\}) \leq 2\sqrt[3]{\epsilon},$$

which implies the assertion of Lemma 2.

Lemma 3. *For any $g \in \mathrm{Diff}_0^3([0,1])$, $\epsilon > 0$, there is $\delta_1 \in (0,1)$ such that the inequality is valid*

$$|\prod_{k=1}^n \frac{g(x_k) - g(x_{k-1})}{(x_k - x_{k-1})\sqrt{g'(x_k) g'(x_{k-1})}} - 1| \leq \epsilon$$

for any natural n and any $\overline{x} = (x_1, ..., x_{n-1}) \in D_n$ satisfying the inequality $\max\limits_{1 \leq k \leq n} (x_k - x_{k-1}) < \delta_1$, where $x_0 = 0$, $x_n = 1$.

Proof. Let $\epsilon \in (0,1)$. Let $C = \max\limits_{t_1,t_2 \in [0,1]} (1 + |\frac{g''(t_1)}{g'(t_1)}| + |\frac{g'''(t_2)}{g'(t_1)}|)^2$, $\delta_1 = \frac{1}{400(C+1)}$, $x_k' = \frac{x_k - x_{k-1}}{2}$ for any k $(1 \leq k \leq n)$.

There are $x_k^*, x_k^{**} x_k^{***} \in (0,1)$ such that

$$g(x_k) - g(x_{k-1}) = g'(x_k')(x_k - x_{k-1}) + \frac{1}{24} g'''(x_k^*)(x_k - x_{k-1})^3 =$$
$$= g'(x_k')(x_k - x_{k-1})(1 + \frac{g'''(x_{k-1}^*)}{24 g'(x_k')}(x_k - x_{k-1})^2),$$
$$g'(x_k) = g'(x_k') + \frac{1}{2} g''(x_k')(x_k - x_{k-1}) + \frac{1}{8} g'''(x_k^{**})(x_k - x_{k-1})^2$$
$$= g'(x_k')(1 + \frac{g''(x_k')}{2g'(x_k')}(x_{k-1} - x_{k-2}) + \frac{g'''(x_k^{**})}{8g'(x_k')}(x_{k-1} - x_{k-2})^2),$$
$$g'(x_{k-1}) = g'(x_k') - \frac{1}{2} g''(x_k')(x_k - x_{k-1}) + \frac{1}{8} g'''(x_k^{***})(x_k - x_{k-1})^2$$
$$= g'(x_k')(1 - \frac{g''(x_k')}{2g'(x_k')}(x_{k-1} - x_{k-2}) + \frac{g'''(x_k^{***})}{8g'(x_k')}(x_{k-1} - x_{k-2})^2).$$

Hence

$$\frac{g(x_k) - g(x_{k-1})}{(x_k - x_{k-1})\sqrt{g'(x_k) g'(x_{k-1})}} = \frac{1 + \lambda_k'(x_k - x_{k-1})}{\sqrt{1 + \lambda_k''(x_k - x_{k-1})}}$$

where

$$\lambda_k' = \frac{g'''(x_{k-1}^*)}{24 g'(x_k')}(x_k - x_{k-1}),$$

$$\lambda_k'' = ((\frac{g''(x_k')}{2g'(x_k')})^2 + \frac{g'''(x_k^{**}) + g'''(x_k^{***})}{8g'(x_k')})(x_k - x_{k-1}) +$$

$$+\frac{g''(x'_k)(g'''(x^{***}_k)-g'''(x^{**}_k))}{16(g'(x'_k))^2}(x_k-x_{k-1})^2+\frac{g'''(x^{***}_k)g'''(x^{**}_k)}{64(g'(x'_k))^2}(x_k-x_{k-1})^3.$$

As $(x_k - x_{k-1}) < \delta_1$, there are $|\lambda'_k| < C\delta_1 < \frac{\epsilon}{100}$, $|\lambda''_k| < C\delta_1 < \frac{\epsilon}{100}$.

We have

$$\sigma = \ln(\prod_{k=1}^{n}\frac{g(x_k)-g(x_{k-1})}{(x_k-x_{k-1})\sqrt{g'(x_k)g'(x_{k-1})}}) =$$

$$=\sum_{k=1}^{n}(\ln(1+\lambda'_k(x_k-x_{k-1})-\frac{1}{2}\ln(1+\lambda''_k(x_k-x_{k-1}))$$

and

$$|\sigma| \le 2\sum_{k=1}^{n}(|\lambda'_k|+|\lambda''_k|)(x_k-x_{k-1}) \le \frac{\epsilon}{10}\sum_{k=1}^{n}(x_k-x_{k-1}) = \frac{\epsilon}{10},$$

therefore

$$|\prod_{k=1}^{n}\frac{g(x_k)-g(x_{k-1})}{(x_k-x_{k-1})\sqrt{g'(x_k)g'(x_{k-1})}}-1| = |e^{\sigma}-1| \le e^{\frac{\epsilon}{10}}-e^{-\frac{\epsilon}{10}} \le \frac{\epsilon}{5}+\frac{\epsilon}{5} < \epsilon,$$

which implies the assertion of Lemma 3.

Introduce the mapping $Q_n : D_n \times E^n_\delta \to E_\delta = \mathrm{Diff}^{1,\delta}_+([0,1])$ by setting $f_n \circ (\tilde{l}_n)^{-1} = Q_n(x_1,...,x_{n-1},\varphi_1,...,\varphi_n)$, where

$$f_n(t) = x_{k-1} + (x_k - x_{k-1})\varphi_k(n(t-\frac{k-1}{n})),$$

$$\tilde{l}_n(t) = \frac{1}{x_1-x_0+\sum\limits_{m=2}^{n}(x_m-x_{m-1})\frac{\varphi'_2(0)\varphi'_3(0)...\varphi'_m(0)}{\varphi'_1(1)\varphi'_2(1)...\varphi'_{m-1}(1)}}\cdot$$

$$\cdot(x_1-x_0+\sum_{m=2}^{k-1}(x_m-x_{m-1})\frac{\varphi'_2(0)\varphi'_3(0)...\varphi'_m(0)}{\varphi'_1(1)\varphi'_2(1)...\varphi'_{m-1}(1)}+$$

$$+(x_k-x_{k-1})\frac{\varphi'_2(0)\varphi'_3(0)...\varphi'_k(0)}{\varphi'_1(1)\varphi'_2(1)...\varphi'_{k-1}(1)}n(t-\frac{k-1}{n}))$$

for $t \in [\frac{k-1}{n}, \frac{k}{n}]$, $(x_1,...,x_{n-1}) \in D_n$, $(\varphi_1,...,\varphi_n) \in E^n_\delta$.

The function $f = f_n \circ (\tilde{l}_n)^{-1}$ belongs to $\mathrm{Diff}^{1,\delta}_+([0,1])$, because the left derivation

$$f'((\tilde{l}_n)^{-1}(\frac{k-1}{n}-0)) = n(x_{k-1}-x_{k-2})\varphi_{k-1}(1)\cdot$$

$$\cdot\frac{x_1-x_0+\sum\limits_{m=2}^{n}(x_m-x_{m-1})\frac{\varphi'_2(0)\varphi'_3(0)...\varphi'_m(0)}{\varphi'_1(1)\varphi'_2(1)...\varphi'_{m-1}(1)}}{(x_{k-1}-x_{k-2})\frac{\varphi'_2(0)\varphi'_3(0)...\varphi'_{k-1}(0)}{\varphi'_1(1)\varphi'_2(1)...\varphi'_{k-2}(1)}n} =$$

$$=(x_1-x_0+\sum_{m=2}^{n}(x_m-x_{m-1})\frac{\varphi'_2(0)\varphi'_3(0)...\varphi'_m(0)}{\varphi'_1(1)\varphi'_2(1)...\varphi'_{m-1}(1)})\frac{\varphi'_1(1)\varphi'_2(1)...\varphi'_{k-1}(1)}{\varphi'_2(0)\varphi'_3(0)...\varphi'_{k-1}(0)}.$$

is equal to the right derivation

$$f'((\tilde{l}_n)^{-1}(\frac{k-1}{n}+0)) = n(x_k-x_{k-1})\varphi_k(0)\cdot$$

$$\cdot\frac{x_1-x_0+\sum\limits_{m=2}^{n}(x_m-x_{m-1})\frac{\varphi_2'(0)\varphi_3'(0)...\varphi_m'(0)}{\varphi_1'(1)\varphi_2'(1)...\varphi_{m-1}'(1)}}{(x_k-x_{k-1})\frac{\varphi_2'(0)\varphi_3'(0)...\varphi_k'(0)}{\varphi_1'(1)\varphi_2'(1)...\varphi_{k-1}'(1)}n}=$$

$$=(x_1-x_0+\sum_{m=2}^{n}(x_m-x_{m-1})\frac{\varphi_2'(0)\varphi_3'(0)...\varphi_m'(0)}{\varphi_1'(1)\varphi_2'(1)...\varphi_{m-1}'(1)})\frac{\varphi_1'(1)\varphi_2'(1)...\varphi_{k-1}'(1)}{\varphi_2'(0)\varphi_3'(0)...\varphi_{k-1}'(0)}.$$

Let a subgroup G of $\mathrm{Diff}_0^3([0,1])$ satisfies condition (a). We write

$$L_{\delta,n}(F)=\int\limits_{D_{l_n}}\int\limits_{E_\delta}...\int\limits_{E_\delta}F(Q_{l_n}(\bar{x},\varphi_1,...,\varphi_{l_n}))\eta_n(d\bar{x})\nu(d\varphi_1)...\nu(d\varphi_{l_n})$$

for any function $F\in C_b(E_\delta)=C_b(\mathrm{Diff}_+^{1,\delta}([0,1]))$.

Theorem 3. *If a subgroup G of $\mathrm{Diff}_0^3([0,1])$ satisfies condition (a) then $\lim\limits_{n\to\infty}|L_{\delta,n}(F_g)-L_{\delta,n}(F)|=0$ for any function $F\in C_b(\mathrm{Diff}_+^{1,\delta}([0,1]))$ and any diffeomorphism $g\in G$.*

Proof. Let $F\in C_b(\mathrm{Diff}_+^{1,\delta}([0,1]))$, $g\in G$, $C=\sup\limits_{g\in E_\delta}|F(f)|$.

Let $\epsilon\in(0,1)$.

It follows from Lemma 3 that it exists $\delta_1\in(0,1)$ such that

$$|\prod_{k=1}^{n}\frac{g(x_k)-g(x_{k-1})}{(x_k-x_{k-1})\sqrt{g'(x_k)g'(x_{k-1})}}-1|\le\epsilon$$

for any positive integer n and for $\overline{x}=(x_1,...,x_{n-1})\in D_n$ satisfying the inequalities $\max\limits_{1\le k\le n}(x_k-x_{k-1})<\delta_1$.

Let us take positive ϵ_1 satisfying the inequalities $\epsilon_1<\frac{1}{8}\epsilon^3$, $\epsilon_1<\delta_1$, $e^{4c_5C_g\sqrt[3]{\epsilon_1}}-e^{-4c_4C_g\sqrt[3]{\epsilon_1}}<\epsilon$.

It follows from Lemma 2 that the inequality is valid $\nu_n(E_\delta^n\setminus X_{\sqrt[3]{\epsilon_1},g,\overline{x}})\le 2\sqrt[3]{\epsilon_1}\le\epsilon$ for any positive integer n and for any $\overline{x}=(x_1,...,x_{n-1})\in D_n$ satisfying the inequalities $\max\limits_{1\le k\le n}(x_k-x_{k-1})<\epsilon_1$.

Since the subgroup G satisfies condition (a) we have that

(i) there are a integer $l_n>l_{n-1}\,(l_0=1)$ for any natural n and a countably additive Borel measure η_n on D_{l_n} such that $\eta_n(D_{l_n})=1$,

(ii) we can find natural $N(\epsilon_1,g)$ such that, for any $n>N(\epsilon_1,g)$, it exists a Borel subset $Z_{n,\epsilon_1,g}\subset D_{l_n}$ that $\eta_n(Z_{n,\epsilon_1,g})>1-\epsilon_1$ and $\max\limits_{1\le k\le l_n}(x_k-x_{k-1})<\epsilon_1$ for any $(x_1,x_2,...,x_{l_n-1})\in Z_{n,\epsilon_1,g}$ where $x_0=0,\ x_{l_n}=1$,

(iii) $(1-\epsilon_1)\eta_n(Y)<\eta_n(gY)<(1+\epsilon_1)\eta_n(Y)$ for any Borel subset $Y\subset Z_{n,\varepsilon,g}$ where $gY=\{(g(x_1),g(x_2),...,g(x_{l_n-1})):(x_1,x_2,...,x_{l_n-1})\in Y\}$.

Hence it exists the function $\varrho_n:Z_{n,\epsilon_1,g}\to\mathbf{R}$ such that $1-\epsilon_1\le\varrho_n(\overline{x})\le1+\epsilon_1$ for any $\overline{x}\in Z_{n,\epsilon_1,g}$, $\eta_n(gY)=\int\limits_Y\varrho_n(\overline{x})\eta_n(d\overline{x})$ for any Borel subset $Y\subset Z_{n,\varepsilon,g}$.

Let $y_k=g(x_k)$, $\overline{y}=(y_1,...,y_{l_n-1})\in D_{l_n}$, $g^{-1}(\overline{y})=(x_1,...,x_{l_n-1})\in D_{l_n}$. We receive

$$gX_{\sqrt[3]{\epsilon_1},g,g^{-1}(\overline{y})}=\{(\varphi_1,...,\varphi_{l_n}):(q_1,...,q_{l_n})\in X_{\sqrt[3]{\epsilon_1},g,\overline{x}},$$

$$Q_{l_n}(y_1,...,y_{l_n-1},\varphi_1,...,\varphi_n)=g\circ(Q_{l_n}(x_1,x_2,...,x_{l_n-1},q_1,...,q_{l_n}))\}.$$

It is easy to see that $\varphi_k(t)=\frac{g(x_{k-1}+(x_k-x_{k-1})q_k(t))-g(x_{k-1})}{g(x_k)-g(x_{k-1})}$, because

$$\frac{(y_k-y_{k-1})\varphi_2'(0)\varphi_3'(0)...\varphi_k'(0)}{(y_1-y_0)\varphi_1'(1)\varphi_2'(1)...\varphi_{k-1}'(1)}=\frac{(x_k-x_{k-1})q_2'(0)q_3'(0)...q_k'(0)}{(x_1-x_0)q_1'(1)q_2'(1)...q_{k-1}'(1)}$$

We have

$$\int\limits_{gZ_{n,\epsilon_1,g}}\nu_{l_n}(gX_{\sqrt[3]{\epsilon_1},g,g^{-1}(\overline{y})})\eta_n(d\overline{y})=$$

$$=\int\limits_{Z_{n,\epsilon_1,g}}(\int\limits_{X_{\sqrt[3]{\epsilon_1},g,\overline{x}}}\exp(\sum_{k=1}^{l_n}[(x_k-x_{k-1})(\frac{g''(x_{k-1})}{g'(x_{k-1})}q_k'(0)-\frac{g''(x_k)}{g'(x_k)}q_k'(1))+$$

$$+(x_k-x_{k-1})^2\int_0^1 S_g(x_{k-1}+(x_k-x_{k-1})q_k(t))(q_k'(t))^2dt])\nu(dq_1)...\nu(dq_{l_n}))$$

$$\varrho_n(\overline{x})\prod_{k=1}^{l_n}\frac{g(x_k)-g(x_{k-1})}{(x_k-x_{k-1})\sqrt{g'(x_k)g'(x_{k-1})}}\eta_n(d\overline{x})\geq$$

$$\geq(1-\epsilon)^3\int\limits_{Z_{n,\epsilon_1,g}}\nu_{l_n}(X_{\sqrt[3]{\epsilon_1},g,\overline{x}})\eta_n(d\overline{x})\geq(1-\epsilon)^5.$$

Hence,

$$|L_{\delta,n}(F_g)-\int\limits_{gZ_{n,\epsilon_1,g}}(\int\limits_{gX_{\sqrt[3]{\epsilon_1},g,g^{-1}(\overline{y})}}F_g(Q_{l_n}(\overline{y},\varphi_1,...,\varphi_{l_n}))$$

$$\nu(d\varphi_1)...\nu(d\varphi_{l_n}))\eta_n(d\overline{y})|\leq C(1-(1-\epsilon)^5)$$

and

$$|L_{\delta,n}(F)-\int\limits_{Z_{n,\epsilon_1,g}}(\int\limits_{X_{\sqrt[3]{\epsilon_1},g,\overline{x}}}F(Q_{l_n}(\overline{x},q_1,...,q_{l_n}))$$

$$\nu(dq_1)...\nu(dq_{l_n}))\eta_n(d\overline{x})|\leq C(1-(1-\epsilon)^2).$$

We have

$$|\int\limits_{gZ_{n,\epsilon_1,g}}(\int\limits_{gX_{\sqrt[3]{\epsilon_1},g,g^{-1}(\overline{y})}}F_g(Q_{l_n}(\overline{y},\varphi_1,...,\varphi_{l_n}))$$

$$\nu(d\varphi_1)...\nu(d\varphi_{l_n}))\eta_n(d\overline{y})-$$

$$-\int\limits_{Z_{n,\epsilon_1,g}}(\int\limits_{X_{\sqrt[3]{\epsilon_1},g,\overline{x}}}F(Q_{l_n}(\overline{x},q_1,...,q_{l_n}))$$

$$\nu(dq_1)...\nu(dq_{l_n}))\eta_n(d\overline{x})|\leq$$

$$\leq\int\limits_{Z_{n,\epsilon_1,g}}(\int\limits_{X_{\sqrt[3]{\epsilon_1},g,\overline{x}}}|\exp(\sum_{k=1}^{n}[(x_k-x_{k-1})(\frac{g''(x_{k-1})}{g'(x_{k-1})}q_k'(0)-\frac{g''(x_k)}{g'(x_k)}q_k'(1))+$$

$$+(x_k-x_{k-1})^2\int_0^1 S_g(x_{k-1}+(x_k-x_{k-1})q_k(t))(q_k'(t))^2dt])$$

$$\varrho_n(\overline{x})\prod_{k=1}^{l_n}\frac{g(x_k)-g(x_{k-1})}{(x_k-x_{k-1})\sqrt{g'(x_k)g'(x_{k-1})}}-1|$$

$$|F(Q_{l_n}(\overline{x},q_1,...,q_{l_n}))|\nu(dq_1)...\nu(dq_n))\eta_n(d\overline{x})\leq$$

$$\leq C\epsilon(2+\epsilon)\int\limits_{Z_{n,\epsilon_1,g}}\nu_n(X_{\sqrt[3]{\epsilon_1},g,\overline{x}})\eta_n(d\overline{x})\leq C\epsilon(2+\epsilon),$$

which implies the assertion of Theorem 3.

Define a ultrafilter $\Im$ on the set positive integers such that $\Im$ contains the sets $\{n, n+1, ...\}$ for any positive integer n. We set $L_\delta(F) = \lim\limits_{\Im} L_{\delta,n}(F)$ for any function $F \in C_b(E_\delta)$.

Note that the limit always exists because $|L_{\delta,n}(F)| \leq \sup\limits_{f\in E_\delta} |F(f)|$.

It is easy to see that $L(e_{1,\delta}) = 1$, $|L_\delta(F)| \leq \sup\limits_{f\in E_\delta} |F(f)|$, and $L(F) \geq 0$ for any nonnegative function $F \in C_b(\mathrm{Diff}_+^{1,\delta}([0,1]))$. In turn, Theorem 1 follows from Theorem 3.

2. Proof of Theorem 2.

Let $B(G)$ be the linear space of all bounded functions on the group G.

Let positive $\delta < \frac{1}{2}$, let

$$p_\delta(f) = |\ln(f'(0)| + \sup_{t_1,t_2\in[0,1]} \frac{|\ln(f'(t_2)) - \ln(f'(t_1))|}{|t_2-t_1|^\delta}$$

and $r(f) = \inf\limits_{h\in G} p_\delta(h^{-1}\circ f)$ for $f \in \mathrm{Diff}_+^{1,\delta}([0,1])$, $\theta(t) = 1-t$ for $t \in [0,1]$ and $\theta(t) = 0$ for $t > 1$.

For any fixed $f \in \mathrm{Diff}_+^{1,\delta}([0,1])$, $C > 0$, the set of functions $\{\psi : \psi(t) = \ln(g'(t)),\ g \in G,\ \ p_\delta(g\circ f) < C\}$ contain in a compact subset of the space $C([0,1])$, therefore it is finite according to condition (a). Hence, we can define the linear mapping $\pi_\delta : B(G) \to C_b(\mathrm{Diff}_+^{1,\delta}([0,1]))$ by setting

$$\pi_\delta F(f) = \frac{\sum\limits_{h\in G}\theta(p_\delta(h^{-1}\circ f) - r(f))F(h)}{\sum\limits_{h\in G}\theta(p_\delta(h^{-1}\circ f) - r(f))}.$$

Assign a linear functional $l : B(G) \to \mathbf{R}$ by setting $l(F) = L_\delta(\pi_\delta F)$.

It is easy to see that

$$|l(F)| = |L_\delta(\pi_\delta F)| \leq \sup_{f\in\mathrm{Diff}_+^{1,\delta}([0,1])} |\pi_\delta F(f)| \leq \sup_{g\in G}|F(g)|,$$

$l(F) \geq 0$ for any nonnegative function $F \in B(G)$, and $l(e_G) = 1$, where $e_G(g) = 1$ for all $g \in G$.

Denote by $F_g(h) = F(g^{-1}\circ h)$ for $F \in B(G)$, $g, h \in G$.

We have

$$\pi_\delta F_g(f) = \frac{\sum\limits_{h\in G}\theta(p_\delta(h^{-1}\circ f) - r(f))F(g^{-1}\circ h)}{\sum\limits_{h\in G}\theta(p_\delta(h^{-1}\circ f) - r(f))} =$$

$$= \frac{\sum\limits_{h\in G}\theta(p_\delta(h^{-1}\circ g\circ f) - r(g\circ f))F(h)}{\sum\limits_{h\in G}\theta(p_\delta(h^{-1}\circ g\circ f) - r(g\circ f))} = \pi_\delta F(g\circ f),$$

hence $l(F_g) = L_\delta(\pi_\delta F_g) = L_\delta(\pi_\delta F) = l(F)$,
which implies the assertion of Theorem 2.

3. Proof of Corollary 2.1.

Let $f_1(t) = \frac{1}{2}t$ for $0 \le t \le \frac{1}{2}$, $f_1(t) = t - \frac{1}{4}$ for $\frac{1}{2} \le t \le \frac{3}{4}$,
$f_1(t) = 2t - 1$ for $\frac{3}{4} \le t \le 1$ and
$f_2(t) = t$ for $0 \le t \le \frac{1}{2}$, $f_2(t) = \frac{1}{2}t + \frac{1}{4}$ for $\frac{1}{2} \le t \le \frac{3}{4}$,
$f_2(t) = t - \frac{1}{8}$ for $\frac{3}{4} \le t \le \frac{7}{8}$, $f_2(t) = 2t - 1$ for $\frac{7}{8} \le t \le 1$.

The Thompson's group F is generated by f_1 and f_2.

Denote by $r_n = 1 - \frac{1}{2^{n+1}}$ for integer $n \ge 0$ and $r_{-k} = \frac{1}{2^k}$ for integer $k \ge 1$. We have $f_1(r_n) = r_{n-1}$ for any integer n.

The group F act on D_n by $f(x_1, ..., x_{n-1}) = (f(x_1), ..., f(x_{n-1}))$ for any $f \in F$, $(x_1, ..., x_{n-1}) \in D_n$.

Let$I_0^k = \{(r_0, r_1)\}$,

$$I_n^k = \{f_2 f_1^{-l_1} f_2 f_1^{l_1 - l_2} f_2 ... f_1^{l_{n-2} - l_{n-1}} f_2 f_1^{l_{n-1}}(r_0, r_1, ..., r_{n+1}) : \ 0 \le l_i \le \min(k, i)\},$$

$a_{n,k} = |I_n^k|$ for $n \ge 0$, $k \ge 1$.

Lemma 4. $\lim\limits_{n\to\infty} \frac{a_{n,k}}{4^{n+1}\cos^{2n}\frac{\pi}{k+2}} = (k+2)\sin^2\frac{\pi}{k+2}$.

Proof. It is easy to see that $a_{0,k} = 1$, $a_{n,1} = 1$, $a_{n+1,k+1} = \sum\limits_{i=0}^{n} a_{i,k+1} a_{n-i,k}$.

Let $u_k(t) = \sum\limits_{n=0}^{\infty} a_{n,k} t^n$.

We have $u_1(t) = \frac{1}{1-t}$, $u_{k+1}(t) = 1 + t\ u_{k+1}(t) u_k(t)$ or $u_{k+1}(t) = \frac{1}{1 - t u_k(t)}$.

Taking $u_k(t) = \frac{p_{k-1}(t)}{p_k(t)}$, $p_0(t) = 1$, $p_1(t) = 1 - t$ we find

$$u_{k+1}(t) = \frac{1}{1 - t\frac{p_{k-1}(t)}{p_k(t)}} = \frac{p_k(t)}{p_k(t) - t p_{k-1}(t)},$$

$$p_{k+1}(t) = p_k(t) - t p_{k-1}(t).$$

That means

$$p_k(t) = \frac{1}{2^{k+2}\sqrt{1-4t}}[(1 + \sqrt{1-4t})^{k+2} - (1 - \sqrt{1-4t})^{k+2}]$$

or $p_k(t) = \prod\limits_{l=1}^{[\frac{k+1}{2}]} (1 - 4t\cos^2\frac{\pi l}{k+2})$.

Taking $m = [\frac{k+1}{2}]$ we find

$$u_k(t) = \frac{4(k+2)\sin^2\frac{\pi}{k+2}}{1 - 4t\cos^2\frac{\pi}{k+2}} + ... + \frac{4(k+2)\sin^2\frac{\pi m}{k+2}}{1 - 4t\cos^2\frac{\pi m}{k+2}} =$$

$$= \sum_{n=0}^{\infty} 4^{n+1}(k+2)t^n(\sin^2\frac{\pi}{k+2}\cos^{2n}\frac{\pi}{k+2} + ... + \sin^2\frac{\pi m}{k+2}\cos^{2n}\frac{\pi m}{k+2}).$$

Hence

$$a_{n,k} = 4^{n+1}(k+2)(\sin^2\frac{\pi}{k+2}\cos^{2n}\frac{\pi}{k+2} + ... + \sin^2\frac{\pi m}{k+2}\cos^{2n}\frac{\pi m}{k+2})$$

and $\lim\limits_{n\to\infty} \frac{a_{n,k}}{4^{n+1}\cos^{2n}\frac{\pi}{k+2}} = (k+2)\sin^2\frac{\pi}{k+2}$ which implies the assertion of Lemma 4.

For any integer $l \geq 1$, $n_1 \geq 0$, $n_2 \geq 0$,..., $n_l \geq 0$, we write

$$Y_{l,n_1,n_2,...,n_l} = \{(r_0, t_{1,1}, t_{1,2}, ..., t_{1,n_1}, r_1, t_{2,1}, t_{2,2}, ..., t_{2,n_2}, r_2, ...,$$

$$r_{l-1}, t_{l,1}, t_{l,2}, ..., t_{l,n_l}, r_l) : \quad (r_{i-1}, t_{i,1}, t_{i,2}, ..., t_{i,n_i}, r_i) \in f_1^i(I_{n_i}^{l-i}), \quad 0 \leq i \leq l\},$$

$$Y^{l,n} = \bigcup_{n_1+...+n_l=n, n_1\geq 0,...,n_l\geq 0} Y_{l,n_1,...,n_l},$$

$$Y_0^{l,n} = \bigcup_{n_2+...+n_l=n, n_2\geq 0,...,n_l\geq 0} Y_{l,0,n_2,...,n_l}.$$

It is easy to see that $f_2(Y^{l+1,n}) = Y^{l,n+1} \setminus Y_0^{l,n+1}$.

Introduce the mapping $\kappa_n : D_n \to D_{2n}$ by setting

$$\kappa_n(x_1, x_2, ..., x_{n-1}) = (\frac{x_1}{2}, x_1, \frac{x_1+x_2}{2}, x_2, \frac{x_2+x_3}{2}, ..., \frac{x_{n-2}+x_{n-1}}{2}, x_{n-1}, \frac{x_{n-1}+1}{2}).$$

Denote by $X^{0,l,n} = \bigcup\limits_{i=0}^{l} \bigcup\limits_{j=0}^{l-1} f_1^j(Y^{2l-i,n+i})$,

$$X^{m,l,n} = \kappa_{2^{m-1}(n+2l+2)}(\kappa_{2^{m-2}(n+2l+2)}(...\kappa_{2(n+2l+2)}(\kappa_{n+2l+2}(X^{0,l,n}))...)).$$

We have $X^{m,l,n} = \kappa_{2^{m-1}(n+2l+2)}(X^{m-1,l,n})$ and $f_1(\kappa_{2^{m-1}(n+2l+2)}(\overline{x})) = \kappa_{2^{m-1}(n+2l+2)}(f_1(\overline{x}))$, $f_2(\kappa_{2^{m-1}(n+2l+2)}(\overline{x})) = \kappa_{2^{m-1}(n+2l+2)}(f_2(\overline{x}))$ for any $\overline{x} \in X^{m-1,l,n}$.

Also, if $(t_1, t_2, ..., t_{2^m(n+2l+2)})$ belongs to $X^{m,l,n}$ then

$$\{t_1, t_2, ..., t_{2^m(n+2l+2)}\} \supset \{\frac{1}{2^m}, \frac{2}{2^m}, \frac{3}{2^m}, ..., \frac{2^m-1}{2^m}\}$$

for any $m \geq 1$.

Lemma 5. *For any positive ε, there are positive integer l, n such that* $\frac{|f_1(X^{m,l,n})\bigcap X^{m,l,n}|}{|X^{m,l,n}|} > 1-\varepsilon$, $\frac{|f_2(X^{m,l,n})\bigcap X^{m,l,n}|}{|X^{m,l,n}|} > 1-\varepsilon$ *for any $m \geq 0$.*

Proof. It is sufficient to prove for $m = 0$. Take integer $l > \frac{4}{\varepsilon}$.

It follows from Lemma 4 that it exists such integer n that $\frac{|Y_0^{2l-i,n+i}|}{|Y^{2l-i,n+i}|} < \frac{1}{l^3}$ for any $0 \leq i \leq l$.

As $f_2(Y^{2l-i,n+i}) = Y^{l-i-1,n+i+1} \setminus Y_0^{l-i-1,n+i+1}$ we have

$$(1 - \frac{1}{l^2})|Y^{l,n+l}| \leq |Y^{2l,n}| \leq |Y^{2l-1,n+2}| \leq ... \leq |Y^{l,n+l}|$$

and $\frac{|Y^{l,n+l}|}{|\bigcup\limits_{i=0}^{l} Y^{2l-i,n+i}|} < \frac{1}{l}$.

Hence, $\frac{|f_2(X^{0,l,n})\bigcap X^{0,l,n}|}{|X^{0,l,n}|} > 1 - \frac{|Y^{l,n+l}|}{|\bigcup\limits_{i=0}^{l} Y^{2l-i,n+i}|} > 1 - \frac{1}{l} > 1 - \varepsilon$.

As $f_1(\bigcup\limits_{j=0}^{l-1} f_1^j(Y^{2l-i,n+i})) \bigcap \bigcup\limits_{j=0}^{l-1} f_1^j(Y^{2l-i,n+i}) = \bigcup\limits_{j=1}^{l-1} f_1^j(Y^{2l-i,n+i})$

we find $\frac{|f_1(X^{0,l,n})\bigcap X^{0,l,n}|}{|X^{0,l,n}|} = 1 - \frac{1}{l} > 1 - \varepsilon$ which implies the assertion of Lemma 5.

Take a infinite differential function $\psi : \mathbf{R} \to \mathbf{R}$ such that $\psi(t+1) = \psi(t) + 2$, $0 < \psi'(t) \leq 3$ for any $t \in \mathbf{R}$, $\psi'(t) = 3$ for any $t \in [\frac{1}{4}, \frac{3}{4}]$, $\psi(0) = 0$, $\psi(\frac{1}{4}) = \frac{1}{4}$, $\psi'(0) = 1$,$\psi^{(n)}(0) = 0$ for any $n \geq 2$.

For any dyadic rational $r = \frac{k}{2^p} \in (0, 1)$, denote $x_r = \psi^{-p}(k)$, $x'_r = \psi^{-p}(k - \frac{1}{4})$, $x''_r = \psi^{-p}(k + \frac{1}{4})$, $\phi_r(t) = \psi^{-p}(k + t)$.

Let $g_1(t) = \psi^{-1}(t)$ for $0 \le t \le x_{\frac{1}{2}} = \frac{1}{2}$,
$g_1(t) = \psi^{-2}(\psi^2(t) - 1)$ for $x_{\frac{1}{2}} \le t \le x_{\frac{3}{4}}$,
$g_1(t) = \psi(t) - 1$ for $x_{\frac{3}{4}} \le t \le 1$ and
$g_2(t) = t$ for $0 \le t \le x_{\frac{1}{2}}$,
$g_2(t) = \psi^{-2}(\psi(t) + 1)$ for $x_{\frac{1}{2}} \le t \le x_{\frac{3}{4}}$,
$g_2(t) = \psi^{-3}(\psi^3(t) - 1)$ for $x_{\frac{3}{4}} \le t \le x_{\frac{7}{8}}$,
$g_2(t) = \psi(t) - 1$ for $x_{\frac{7}{8}} \le t \le 1$.

In [2] È.Ghys and V.Sergiescu proved that the Thompson's group F is isomorphic to a discrete subgroup G of $\mathrm{Diff}_0^3([0,1])$ which is generated by $\{g_1, g_2\}$ and satisfies condition (b).

Lemma 6. *For any dyadic rational $r \in (0,1)$, there are positive integer $\alpha_1, \alpha_2, \beta_1, \beta_2$ such that $|\alpha_1| \le 1$, $|\alpha_2| \le 1$, $|\beta_1| \le 1$, $|\beta_2| \le 1$,*
$g_1(\phi_r(t)) = \phi_{f_1(r)}(\psi^{\alpha_1}(t))$, $g_2(\phi_r(t)) = \phi_{f_2(r)}(\psi^{\beta_1}(t))$, $g_1(\phi_r(-t)) = \phi_{f_1(r)}(\psi^{\alpha_2}(-t))$, $g_2(\phi_r(-t)) = \phi_{f_2(r)}(\psi^{\beta_2}(-t))$ for any $t \in [0, \frac{1}{4}]$.

Proof. Let $t \in [0, \frac{1}{4}]$.

If $r = \frac{1}{2}$ we have $f_1(r) = \frac{1}{4}$, $f_2(r) = \frac{1}{2}$,

$$g_1(\phi_r(t)) = \psi^{-2}(\psi^2(\psi^{-1}(t+1)) - 1) = \psi^{-2}(\psi(t) + 1) = \phi_{f_1(r)}(\psi(t)),$$

$$g_2(\phi_r(t)) = \psi^{-2}(\psi((\psi^{-1}(t+1))) + 1) = \psi^{-1}((\psi^{-1}(t) + 1) = \phi_{f_2(r)}(\psi^{-1}(t)),$$

$$g_1(\phi_r(-t)) = \psi^{-1}(\psi^{-1}(-t+1)) = \psi^{-2}(-t+1) = \phi_{f_1(r)}(-t),$$

$$g_2(\phi_r(-t)) = \psi^{-1}(-t+1) = \phi_{f_2(r)}(-t).$$

Hence $\alpha_1 = 1, \alpha_2 = 0, \beta_1 = -1, \beta_2 = 0$.

If $r = \frac{3}{4}$ we have $f_1(r) = \frac{1}{2}$, $f_2(r) = \frac{5}{8}$,

$$g_1(\phi_r(t)) = \psi(\psi^{-2}(t+3) - 1 = \psi^{-1}(t+1) = \phi_{f_1(r)}(t),$$

$$g_2(\phi_r(t)) = \psi^{-3}(\psi^3(\psi^{-2}(t+3)) - 1) = \psi^{-3}((\psi(t) + 5) = \phi_{f_2(r)}(\psi(t)),$$

$$g_1(\phi_r(-t)) = \psi^{-2}(\psi^2(\psi^{-2}(-t+3)) - 1) = \psi^{-1}(\psi^{-1}(-t) + 1) = \phi_{f_1(r)}(\psi^{-1}(-t)),$$

$$g_2(\phi_r(-t)) = \psi^{-2}(\psi(\psi^{-2}(-t+3)) + 1) = \psi^{-3}(-t+5) = \phi_{f_2(r)}(-t).$$

Hence $\alpha_1 = 0, \alpha_2 = -1, \beta_1 = 1, \beta_2 = 0$.

If $r = \frac{7}{8}$ we have $f_2(r) = \frac{3}{4}$,

$$g_2(\phi_r(t)) = \psi(\psi^{-3}(t+7)) - 1 = \psi^{-2}(t+3) = \phi_{f_2(r)}(t),$$

$$g_2(\phi_r(-t)) = \psi^{-3}(\psi^3(\psi^{-3}(-t+7)) - 1) = \psi^{-2}(\psi^{-1}(-t) + 3) = \phi_{f_2(r)}(-t).$$

Hence $\beta_1 = 0, \beta_2 = -1$.

If $0 < r = \frac{k}{2^p} < \frac{1}{2}$ we have $f_1(r) = \frac{k}{2^{p+1}}$, $f_2(r) = \frac{k}{2^p}$,

$$g_1(\phi_r(\pm t)) = \psi^{-1}(\psi^{-p}(\pm t + k)) = \psi^{-p-1}(\pm t + k) = \phi_{f_1(r)}(\pm t),$$

$$g_2(\phi_r(-t)) = \psi^{-p}(\pm t + k) = \phi_{f_2(r)}(\pm t).$$

Hence $\alpha_1 = \alpha_2 = 0, \beta_1 = \beta_2 = 0$.

If $\frac{1}{2} < r = \frac{k}{2^p} < \frac{3}{4}$ we have $f_1(r) = \frac{k - 2^{p-2}}{2^p}$, $f_2(r) = \frac{k + 2^{p-1}}{2^{p+1}}$,

$$g_1(\phi_r(\pm t)) = \psi^{-2}(\psi^2(\psi^{-p}(\pm t + k)) - 1) = \psi^{-p}(\pm t + k - 2^{p-2}) = \phi_{f_1(r)}(\pm t),$$

$$g_2(\phi_r(\pm t)) = \psi^{-2}(\psi(\psi^{-p}(\pm t + k)) + 1) = \psi^{-p-1}(\pm t + k + 2^{p-1}) = \phi_{f_2(r)}(\pm t).$$

Hence $\alpha_1 = \alpha_2 = 0, \beta_1 = \beta_2 = 0$.

If $\frac{3}{4} < r = \frac{k}{2^p} < 1$ we have $f_1(r) = \frac{k-2^{p-1}}{2^{p-1}}$,

$$g_1(\phi_r(\pm t)) = \psi(\psi^{-p}(\pm t + k)) - 1 = \psi^{-p+1}(\pm t + k - 2^{p-1}) = \phi_{f_1(r)}(\pm t).$$

Hence $\alpha_1 = \alpha_2 = 0$.

If $\frac{3}{4} < r = \frac{k}{2^p} < \frac{7}{8}$ we have $f_2(r) = \frac{k-2^{p-3}}{2^p}$,

$$g_2(\phi_r(\pm t)) = \psi^{-3}(\psi^3(\psi^{-p}(\pm t + k)) - 1) = \psi^{-p}(\pm t + k - 2^{p-3}) = \phi_{f_2(r)}(\pm t).$$

Hence $\beta_1 = \beta_2 = 0$.

If $\frac{7}{8} < r = \frac{k}{2^p} < 1$ we have $f_2(r) = \frac{k-2^{p-1}}{2^{p-1}}$,

$$g_2(\phi_r(\pm t)) = \psi(\psi^{-p}(\pm t + k)) - 1 = \psi^{-p+1}(\pm t + k - 2^{p-1}) = \phi_{f_2(r)}(\pm t).$$

Hence $\beta_1 = \beta_2 = 0$.

Thus, we prove Lemma 6.

Lemma 7. *For any positive ε, there are positive integer N and a finite subset $Z \subset D_N$ such that $\frac{|g_1(Z) \bigcap Z|}{|Z|} > 1 - \varepsilon$, $\frac{|g_2(Z) \bigcap Z|}{|Z|} > 1 - \varepsilon$, $\max\limits_{1 \le k \le N} (x_k - x_{k-1}) < \varepsilon$ for any $(x_1, x_2, ..., x_{N-1}) \in Z$ where $x_0 = 0,\ x_N = 1$.*

Proof. Let $\varepsilon \in (0,1)$.

As $\lim\limits_{m\to\infty} \sum\limits_{l=1}^{2^m-1} (x''_{\frac{l}{2^m}} - x'_{\frac{l}{2^m}}) = \frac{1}{2}$ it exists such $m \ge 1$ that $\max\limits_{1 \le l \le 2^m} (x'_{\frac{l}{2^m}} - x''_{\frac{l-1}{2^m}}) < \varepsilon$, where $x''_0 = \frac{1}{4}$, $x'_1 = \frac{3}{4}$.

By Lemma 5 we find positive integer l, n such that $\frac{|f_1(X^{m,l,n}) \bigcap X^{m,l,n}|}{|X^{m,l,n}|} > 1 - \frac{1}{4}\varepsilon$, $\frac{|f_2(X^{m,l,n}) \bigcap X^{m,l,n}|}{|X^{m,l,n}|} > 1 - \frac{1}{4}\varepsilon$.

Let $k = 2^m(n+2l+2)$, $V_{\overline{t}} = \{t_1, t_2, ..., t_{k-1}\}$ for any $\overline{t} = (t_1, t_2, ..., t_{k-1}) \in X^{m,l,n}$, and $W = \bigcup\limits_{\overline{t} \in X^{m,l,n}} V_{\overline{t}}$.

Take integer $J > \frac{16(k+1)}{\varepsilon}$. Let

$$C = \max_{0 \le j \le J} \Big(\max_{-\frac{1}{4} \le x \le \frac{1}{4}} \big(\max_{r \in W} |(\phi_r(\psi^j(x)))'| + |(\psi^j(x))'|\big)\Big).$$

Take integer $p > \frac{C+1}{\varepsilon}$. Let $N = k(2p+1)$,

$$\begin{gathered} Z = \{(\psi^{j_1}(\frac{1}{4p}), \psi^{j_1}(\frac{2}{4p}), ..., \psi^{j_1}(\frac{p-1}{4p}), \frac{1}{4}, x'_{t_1}, \phi_{t_1}(\psi^{j_2}(-\frac{p-1}{4p})), \phi_{t_1}(\psi^{j_2}(-\frac{p-2}{4p})), ..., \\ \phi_{t_1}(\psi^{j_2}(-\frac{1}{4p})), x_{t_1}, \phi_{t_1}(\psi^{j_3}(\frac{1}{4p})), \phi_{t_1}(\psi^{j_3}(\frac{2}{4p})), ..., \phi_{t_1}(\psi^{j_3}(\frac{p-1}{4p})), x''_{t_1}, \\ x'_{t_2}, \phi_{t_2}(\psi^{j_4}(-\frac{p-1}{4p})), \phi_{t_2}(\psi^{j_4}(-\frac{p-2}{4p})), ..., \phi_{t_2}(\psi^{j_4}(-\frac{1}{4p})), \\ x_{t_2}, \phi_{t_2}(\psi^{j_5}(\frac{1}{4p})), \phi_{t_2}(\psi^{j_5}(\frac{2}{4p})), ..., \phi_{t_2}(\psi^{j_5}(\frac{p-1}{4p})), x''_{t_2}, ..., \\ x'_{t_{k-1}}, \phi_{t_{k-1}}(\psi^{j_{2k-2}}(-\frac{p-1}{4p})), \phi_{t_{k-1}}(\psi^{j_{2k-2}}(-\frac{p-2}{4p})), ..., \phi_{t_{k-1}}(\psi^{j_{2k-2}}(-\frac{1}{4p})), \\ x_{t_{k-1}}, \phi_{t_{k-1}}(\psi^{j_{2k-1}}(\frac{1}{4p})), \phi_{t_{k-1}}(\psi^{j_{2k-1}}(\frac{2}{4p})), ..., \phi_{t_{k-1}}(\psi^{j_{2k-1}}(\frac{p-1}{4p})), x''_{t_{k-1}}, ..., \\ \frac{3}{4}, 1-\psi^{j_{2k}}(-\frac{p-1}{4p}), 1-\psi^{j_{2k}}(-\frac{p-2}{4p}), ..., 1-\psi^{j_{2k}}(-\frac{1}{4p})) : \ \ 0 \le j_1 \le J,\ \ 0 \le j_2 \le J, \\ 0 \le j_3 \le J,\ \ 0 \le j_4 \le J,\ \ 0 \le j_5 \le J, ...,\ \ 0 \le j_{2k-2} \le J,\ \ 0 \le j_{2k-1} \le J, \\ 0 \le j_{2k} \le J,\ \ (t_1, t_2, ..., t_{k-1}) \in X^{m,l,n}\}, \end{gathered}$$

and

$$Z_i=\{(\psi^{j_1}(\frac{1}{4p}),\psi^{j_1}(\frac{2}{4p}),...,\psi^{j_1}(\frac{p-1}{4p}),\frac{1}{4},x'_{t_1},\phi_{t_1}(\psi^{j_2}(-\frac{p-1}{4p})),\phi_{t_1}(\psi^{j_2}(-\frac{p-2}{4p})),...,$$
$$\phi_{t_1}(\psi^{j_2}(-\frac{1}{4p})),x_{t_1},\phi_{t_1}(\psi^{j_3}(\frac{1}{4p})),\phi_{t_1}(\psi^{j_3}(\frac{2}{4p})),...,\phi_{t_1}(\psi^{j_3}(\frac{p-1}{4p})),x''_{t_1},$$
$$x'_{t_2},\phi_{t_2}(\psi^{j_4}(-\frac{p-1}{4p})),\phi_{t_2}(\psi^{j_4}(-\frac{p-2}{4p})),...,\phi_{t_2}(\psi^{j_4}(-\frac{1}{4p})),$$
$$x_{t_2},\phi_{t_2}(\psi^{j_5}(\frac{1}{4p})),\phi_{t_2}(\psi^{j_5}(\frac{2}{4p})),...,\phi_{t_2}(\psi^{j_5}(\frac{p-1}{4p})),x''_{t_2},...,$$
$$x'_{t_{k-1}},\phi_{t_{k-1}}(\psi^{j_{2k-2}}(-\frac{p-1}{4p})),\phi_{t_{k-1}}(\psi^{j_{2k-2}}(-\frac{p-2}{4p})),...,\phi_{t_{k-1}}(\psi^{j_{2k-2}}(-\frac{1}{4p})),$$
$$x_{t_{k-1}},\phi_{t_{k-1}}(\psi^{j_{2k-1}}(\frac{1}{4p})),\phi_{t_{k-1}}(\psi^{j_{2k-1}}(\frac{2}{4p})),...,\phi_{t_{k-1}}(\psi^{j_{2k-1}}(\frac{p-1}{4p})),x''_{t_{k-1}},...,$$
$$\frac{3}{4},1-\psi^{j_{2k}}(-\frac{p-1}{4p}),1-\psi^{j_{2k}}(-\frac{p-2}{4p}),...,1-\psi^{j_{2k}}(-\frac{1}{4p})):\ 1\le j_1\le J-1,\ 1\le j_2\le J-1,$$
$$1\le j_3\le J-1,\ 1\le j_4\le J-1,\ 1\le j_5\le J-1,...,\ 1\le j_{2k-2}\le J-1,\ 1\le j_{2k-1}\le J-1,$$
$$0\le j_{2k}\le J,\quad (t_1,t_2,...,t_{k-1})\in f_i(X^{m,l,n})\bigcap X^{m,l,n}\}$$

where $i=1,\quad 2.$

By Lemma 6 we find $g_1(Z_1)\subset Z$ and $g_2(Z_2)\subset Z$. Hence

$$\frac{|g_1(Z)\bigcap Z|}{|Z|}\le\frac{|Z_1|}{|Z|}=\frac{(J-1)^{(2k)}|f_1(X^{m,l,n})\bigcap X^{m,l,n}|}{(J+1)^{(2k)}|X^{m,l,n}|}>$$
$$>(1-\frac{4k}{J+1})(1-\frac{1}{4}\varepsilon)>(1-\frac{1}{4}\varepsilon)^2>1-\varepsilon,$$
$$\frac{|g_2(Z)\bigcap Z|}{|Z|}\le\frac{|Z_2|}{|Z|}=\frac{(J-1)^{(2k)}|f_2(X^{m,l,n})\bigcap X^{m,l,n}|}{(J+1)^{(2k)}|X^{m,l,n}|}>1-\varepsilon.$$

We have $\phi_r(\psi^j(\frac{i}{4p}))-\phi_r(\psi^j(\frac{i-1}{4p}))\le C\frac{1}{4p}<\varepsilon$, $\psi^j(\frac{i}{4p})-\psi^j(\frac{i-1}{4p})\le C\frac{1}{4p}<\varepsilon$ for any $r\in W$, $1\le i\le p$, $1\le j\le J$ that means $\max\limits_{1\le k'\le N}(x_{k'}-x_{k'-1})<\varepsilon$ for any $(x_1,x_2,...,x_{N-1})\in Z$.

Thus, we prove Lemma 7.

In turn, Corollary 2.1 follows from Theorem 2 and Lemma 7.

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